Existence of an optimal solution

$$\min_{x \in \mathbb{R}^n} f_0(x) \quad \text{subject to} \quad \begin{cases} f_i(x) \leq 0 & i = 1, \dots, p \\ f_j(x) = 0 & j = p+1, \dots, m \\ x \in C \end{cases}$$

Thm: Let C be closed, $f_i : \mathbb{R}^n \to \mathbb{R}$ continuous for $i = 0, \dots, m$ and f_0 coercive over

$$S := \{ x \in C \mid f_i(x) \leq 0, f_j(x) = 0 \text{ for } i = 1 \dots p,\ j = p+1, \dots, m \}$$

that is $\forall (x_n)_{n \in \mathbb{N}}$ in S with $\lim_{n \to \infty} \|x_n\| = \infty$ also $\lim_{n \to \infty} |f(x_n)| = \infty$,

and $\inf_{x \in S} f_0(x) \in \mathbb{R}$.

Then (P) has at least an optimal solution.

Proof: • C closed, f_i cont. $\Rightarrow$ S closed

• let $(x_n)_{n \in \mathbb{N}}$ be a sequence in S s.t.

$$f_0(x_n) \xrightarrow[n \to \infty]{} \inf_{x \in S} f_0(x) \in \mathbb{R}$$

• since f_0 is coercive and $(f_0(x_n))_{n \in \mathbb{N}}$ is obviously bounded also $(x_n)_{n \in \mathbb{N}}$ is bounded

• hence, $\exists (x_{n_k})_{k \in \mathbb{N}}$ s.t. $x_{n_k} \xrightarrow[k \to \infty]{} x^* \in S$ since S was closed

• but f_0 cont

$$\Rightarrow \quad f_0(x^*) = \lim_{k \to \infty} f_0(x_{n_k}) = \inf_{x \in S} f_0(x).$$

Hence, $\exists$ solution x^* of (P). $\square$

Thm: Let f_i, $1 \le i \le m$, be convex and x^* is a local min, then:

(i) f_0 convex $\Rightarrow$ local min = global min

(ii) f_0 strictly convex $\Rightarrow$ min. unique.

Proof: i)

- x^* local min. $\Rightarrow \exists \varepsilon > 0$:

$$\forall x \in B_\varepsilon(x^*) \cap S : \; f_0(x^*) \le f(x)$$

- since f_i, $i = 1 \dots m$, are convex also S is convex

- take $y \ne x^*$, $y \in S$, $\lambda \in (0,1]$ s.t.

$$x^* + \lambda (y - x^*) \in B_\varepsilon(x^*)$$

$$\Rightarrow f_0(x^*) \le f(x^* + \lambda(y - x^*))$$
$$\le (1-\lambda) f_0(x^*) + \lambda f_0(y)$$

$$\Rightarrow f_0(x^*) \le f_0(y)$$

ii) Same argument with strict inequalities.

Kor: The hard margin SVM has a unique solution.

Proof: The hard margin SVM is given by

$$\underbrace{\min_{w \in \mathbb{R}^n} \tfrac{1}{2}\|w\|^2}_{=: f_0(w)} \quad \text{subject to} \quad \begin{cases} \forall i = 1 \dots M \\ \underbrace{y^{(i)} w \cdot x^{(i)} \ge 1}_{\Leftrightarrow \; f_i(w) \le 0} \end{cases}$$

for $f_i(w) = y^{(i)} w x^{(i)} - 1$

Now $\partial_w f_0(w) = w$ (gradient)

$\partial_w^2 f_0(w) = \mathbb{1}$ (hessian)

$\Rightarrow f_0$ is strictly convex

Furthermore, f_0 is coercive, $\mathbb{R}^n$ closed and f_i for $1 \le i \le M$ are affine $\Rightarrow$ convex

$\Rightarrow$ Thms above state

$\exists$ x^* local min

and this is unique. $\square$